Chapter 11

Mass Points

A *mass point* is a point assigned a mass. We use mP to denote a point P with mass m. Here, m can be a positive or negative real number. The mass points technique is fascinating because it is so powerful. However, it could backfire if you do not fully understand the principles of mass points.

Theorem 11.1. (Principles of Mass Points)

1. *A system of mass points has a center of mass. This center is unique.*
2. *Given mass points mP and nQ, their center R has a mass of $m+n$. We say that $mP + nQ = (m+n)R$. R lies on line segment PQ if m and n are both positive or both negative. R is on the extension of line segment PQ if one of m and n is positive and the other is negative. Furthermore, $|m| : |n| = RQ : PR$.*
3. *Commutativity: $mP + nQ = nQ + mP$.*
4. *Associativity: The center of mass of a system of mass points does not change if a subsystem of mass points is replaced by its center with its total mass. That is,*

$$mP + nQ + kR = mP + (nQ + kR) = (mP + nQ) + kR.$$

For example, if we have two mass points $1P$ and $1Q$, then R, the midpoint of line segment PQ, is the center of these two mass points. We say that

$$1P + 1Q = 2R.$$

We can also rewrite it as

$$1P = 2R + (-1)Q,$$

and say that P is the center of mass of $2R$ and $(-1)Q$. Note that P is on the extension of line segment QR. As we will see in Section 11.4, negative mass points can be quite useful in solving complicated geometry problems. Throughout this chapter, we will use "mass" and "weight" interchangeably.

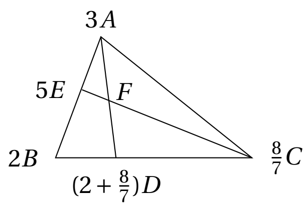
11.1 Mass Points and Cevians

A *cevian* is a line segment from a vertex of a triangle to the other side. The simplest mass points problems are those that involve only cevians. Let's look at an example.

Example 11.1. In $\triangle ABC$, points E and D are selected on sides AB and BC, respectively, such that $AE : EB = 2 : 3$ and $CD : DB = 7 : 4$. Line segment CE intersects AD at F. Find $AF : FD$.

Answer: $22 : 21$

Solution: In order to use E to balance AB, we assign a weight of 3 to A and a weight of 2 to B. To use D as a balance point of BC, we assign a weight of $\frac{8}{7}$ to C.

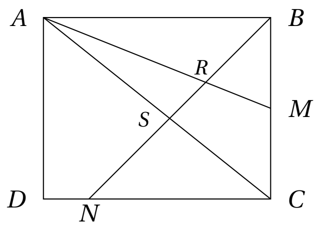


Since the three mass points $3A$, $2B$, and $\frac{8}{7}C$ can be balanced by $3A$ and $(2+\frac{8}{7})D$, the center of mass is on AD. These three mass points can also be balanced by $\frac{8}{7}C$ and $5E$. So, the center of mass is on CE. Therefore, the center of mass is on the intersection of AD and CE, namely, point F. Thus,

$$3AF = (2 + \frac{8}{7})FD,$$
$$AF : FD = \boxed{22 : 21}.$$

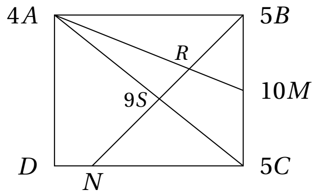
Now let's look at a MathCounts problem.

Example 11.2. In rectangle $ABCD$, point M is the midpoint of side BC, and point N lies on CD such that $DN : NC = 1 : 4$. Line segment BN intersects AM and AC at points R and S, respectively. If $NS : SR : RB = x : y : z$, where x, y, and z are positive integers, what is the minimum possible value of $x + y + z$?



Answer: 126

Solution: We see that $NC : AB = 4 : 5$. Since $\triangle NSC$ and $\triangle BSA$ are similar, $NS : SB = 4 : 5 = CS : SA$. So $NS = \frac{4}{5}SB$. Now let's use mass points in $\triangle ABC$ to find the ratio $SR : SB$. To use S to balance AC, we assign a mass of 5 to C and a mass of 4 to A. To use M to balance CB, we assign a mass of 5 to B. We see that the three mass points $4A$, $5C$, and $5B$ can be balanced by $9S$ and $5B$. They can also be balanced by $4A$ and $10M$. So, the center of mass is the intersection of SB and AM. That is, R is the center of mass of $\triangle ABC$.



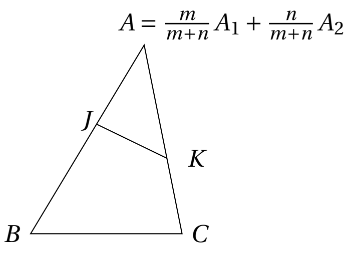
Therefore, $9SR = 5RB$, and so $SR = \frac{5}{9}RB = \frac{5}{5+9}SB = \frac{5}{14}SB$ and $RB = \frac{9}{14}SB$. So,

$$NS : SR : RB = \frac{4}{5} : \frac{5}{14} : \frac{9}{14} = 56 : 25 : 45.$$

Since $\gcd(56, 25, 45) = 1$, $x + y + z$ has a minimum value of $56 + 25 + 45 = \boxed{126}$.

11.2 Mass Points and Transversals

A *transversal* is a line that intersects two other distinct lines. Mass points problems involving transversals in addition to cevians are more complicated than those in Section 11.1. The vertex that is associated with both sides of a transversal will have a *split mass*. For example, in the figure shown, vertex A will have a split mass. To understand this, just imagine that A is actually two points, A_1 and A_2, glued together with A_1B on the left side and A_2C on the right side. If A_1 is assigned a weight of m and A_2 is assigned a weight of n, then $(m+n)A = mA_1 + nA_2$. So $A = \frac{m}{m+n}A_1 + \frac{n}{m+n}A_2$.

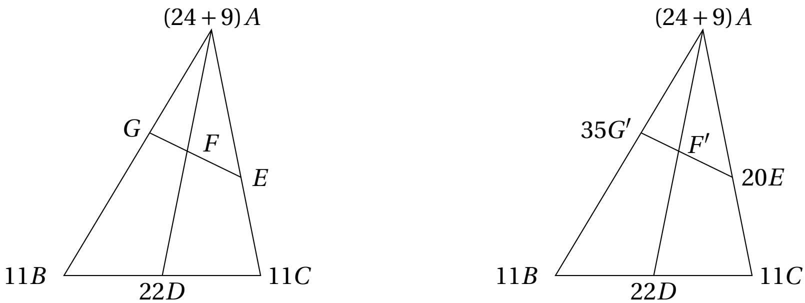


Let's look at the following example.

Example 11.3. In $\triangle ABC$, D is the midpoint of BC. Point E lies on line segment AC such that $AE : EC = 11 : 9$. Point F lies on cevian AD such that $AF : FD = 2 : 3$. The extension of line segment EF intersects AB at G. What is $GF : FE$?

Answer: $4 : 7$

Solution: Since GE is a transversal, we treat vertex A as two vertices, A_1 and A_2, glued together with A_1B on the left side and A_2C on the right side. In order to use E to balance A_2C, we assign a weight of 11 to C and a weight of 9 to A_2. To use D to balance BC, we assign a weight of 11 to B. So D has a weight of 22. To use F to balance AD, A must have a total weight of 33 because $AF : FD = 2 : 3$.



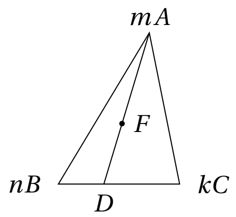
We see that $55F$ is the center of mass of three mass points $33A$, $11B$, and $11C$ because it is the balance point of AD and D is the balance point of BC. Since A_2 has a weight of 9, A_1 must have a weight of 24. Next we pick the point G' on AB such that $AG' : G'B = 11 : 24$. We see that G' is the balance point of A_1 and B. We then connect G' with E and let F' be the intersection of AD and $G'E$.

We see that $24A_1$ and $11B$ are balanced by $35G'$, and $9A_2$ and $11C$ are balanced by $20E$. So $33A$, $11B$, and $11C$ are balanced by $35G'$ and $20E$. Since these three mass points are also balanced by $33A$ and $22D$, the intersection of $G'E$ and AD, namely, F', is the center of mass. Therefore F and F' are the same point. Thus, G and G' are also the same point. So F is the balance point of GE and $GF : FE = 20 : 35 = \boxed{4 : 7}$.

11.3 Multiple Mass Points Systems

Sometimes problems are so involved that it's not enough to assign mass to each vertex once. In such situations, we need to set up mass points systems multiple times. Before we go on to our next example, let's prove the following theorem.

Theorem 11.2. *Let D be a point on side BC of triangle ABC. Let m, n, and k be masses assigned to A, B, and C, respectively. Let F be the center of mass of mA, nB, and kC. Then D is the balance point of nB and kC if and only if F is on AD.*

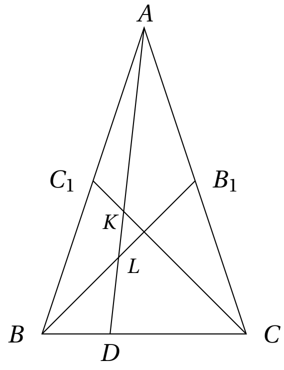


Proof. If D is the balance point of nB and kC, then the three mass points mA, nB, and kC are balanced by mA and $(n+k)D$. So the center of mass must be on AD. Thus F is on AD.

To show that the other direction of the statement is true, we use proof by contradiction. Suppose that F, the center of mass, is on AD. If point D', different from D, is on BC and is the balance point of nB and kC. Then the center of three mass points mA, nB, and kC are balanced by mA and $(n+k)D'$. So F, the center of mass, is on AD'. Since $D \neq D'$, F is not on AD. This is a contradiction. □

Theorem 11.2 comes in handy, as we will see in our next example.

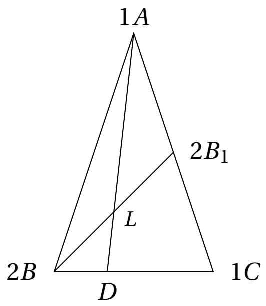
Example 11.4. In $\triangle ABC$, BB_1 and CC_1 are medians. D is on BC such that L, the intersection of AD and BB_1, is the midpoint of BB_1. K is the intersection of AD and CC_1. What is $C_1K : KC$?

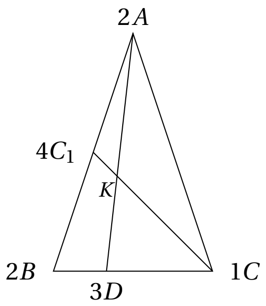


Answer: $\frac{1}{4}$

Solution: To solve this problem, we will first use L as the center of mass of triangle ABC to find the ratio $BD : DC$. Next we will use K as the center of mass to get the ratio $C_1K : KC$.

We assign masses to make L the center of mass of triangle ABC. To use B_1 to balance AC, we let A and C each have a weight of 1. Then B_1 has a weight of 2. To use L to balance BB_1, we assign a weight of 2 to B. We see that L is the center of mass of $1A$, $2B$, and $1C$. By Theorem 11.2, D is the balance point of $2B$ and $1C$ and $BD : DC = 1 : 2$.





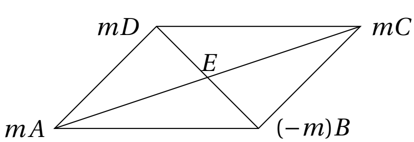
Next we assign masses to make K the center of mass. Since $BD : DC = 1 : 2$, we assign a weight of 2 to B and a weight of 1 to C. So D has a weight of 3. Since C_1 is the midpoint of AB, we assign a weight of 2 to A. We see that the three mass points $2A$, $2B$, and $1C$ are balanced by $2A$ and $3D$. Also they are balanced by $4C_1$ and $1C$. So K, the intersection of CC_1 and AD, is the center of mass. So $4C_1K = 1KC$. Therefore, $\dfrac{C_1K}{KC} = \boxed{\dfrac{1}{4}}$.

11.4 Mass Points in Space

The application of mass points technique is not limited to points in a plane. The principles of mass points apply to mass points in space as well. Negative mass points comes in handy when we try to solve 3-dimensional geometry problems.

Theorem 11.3. *Let $ABCD$ be a parallelogram. A, B, and C are assigned with masses m, $-m$, and m, respectively. Then D is the center of mass of the three mass points mA, $(-m)B$, and mC. That is, $mA + (-m)B + mC = mD$.*

Proof. Let E be the intersection of AC and BD. Since the diagonals of a parallelogram bisect each other, E is the midpoint of AC. So $2mE$ is the balance point of mA and mC.



Since E is also the midpoint of DB, $2mE = mD + mB$. So

$$mD = 2mE + (-m)B = mA + mC + (-m)B = mA + (-m)B + mC. \qquad \square$$

The following example shows how negative mass points can be used to simplify calculations.

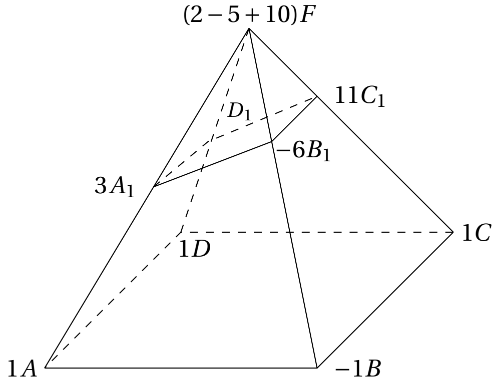
Example 11.5. The base $ABCD$ of a pyramid $FABCD$ is a parallelogram. The plane α intersects AF, BF, CF, and DF at points A_1, B_1, C_1, and D_1, respectively. Given that

$$\frac{AA_1}{A_1F} = 2, \qquad \frac{BB_1}{B_1F} = 5, \qquad \frac{CC_1}{C_1F} = 10,$$

find the ratio $\dfrac{DD_1}{D_1F}$.

Answer: 7

Solution: We would like to use A_1 to balance AF, B_1 to balance BF, and C_1 to balance CF. To make D the center of mass of A, B, and C, we assign a weight of 1 to A, a weight of -1 to B, and a weight of 1 to C. Thus F will have a split mass because $1A$, $(-1)B$, and $1C$ each imposes a weight of 2, -5, and 10 on F, respectively.



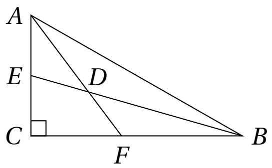
Now $3A_1$ is the balance point of AF; $-6B_1$ is the balance point of BF; and $11C_1$ is the balance point of CF. We see that the four mass points, $1A$, $-1B$, $1C$, and $(2-5+10)F$, are balanced by $3A_1$, $-6B_1$, and $11C_1$. So the center of mass is on the plane α. By Theorem 11.3, $1D$ is the center of mass of $1A$, $-1B$, and $1C$. So the center of mass of $1A$, $-1B$, $1C$, and $(2-5+10)F$ is on DF. Thus, D_1, the intersection of DF and the plane α, is the center of mass. Therefore,

$$\frac{DD_1}{D_1F} = \frac{2-5+10}{1} = \boxed{7}.$$

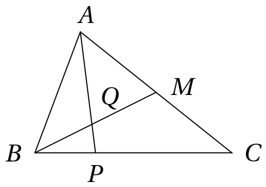
Problems for Chapter 11

Problem 11.1. In $\triangle ABC$, point D is on AC with $AD : DC = 2 : 1$, and point E is on AB with $AE : EB = 2 : 3$. Line segments EC and DB intersect at point K. What is $DK : KB$? (MathCounts)

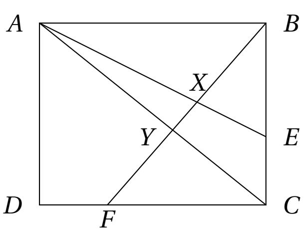
Problem 11.2. In right triangle ABC, $AC = 4$ and $CB = 7$. E is the midpoint of AC. Point F lies on CB such that $CF : FB = 3 : 4$. D is the intersection of AF and EB. What is the area of $\triangle ABD$? (MathCounts)



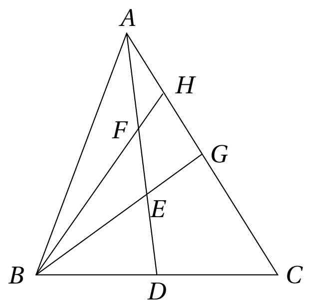
Problem 11.3. Let M and P be points on the sides AC and BC of $\triangle ABC$, respectively, so that $AM : MC = 3 : 1$ and $BP : PC = 1 : 2$. Let Q be the intersection of AP and BM. Given that the area of $\triangle BPQ$ is equal to 1 square inch, find the area of $\triangle ABC$.



Problem 11.4. In rectangle $ABCD$, point E lies on BC so that $\frac{BE}{EC} = 2$ and point F lies on DC so that $\frac{CF}{FD} = 2$. Line segments AE and AC intersect BF at points X and Y, respectively. Given that $FY : YX : XB = a : b : c$, where a, b, and c are relatively prime positive integers, what is the value of $a + b + c$? (MathCounts)

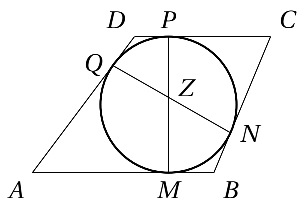


Problem 11.5. Triangle ABC has sides $AB = 39$, $BC = 57$, and $CA = 70$ as shown. Median AD is divided into three congruent segments by points E and F. Lines BE and BF intersect side AC at points G and H, respectively. Find the distance from G to H. (Purple Comet)



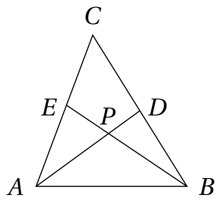
Problem 11.6. The sides of $\triangle ABC$ are $AB = 13$, $BC = 15$, and $AC = 14$. Let BD be an altitude of the triangle. The angle bisector of $\angle C$ intersects side AB at F and altitude BD at E. Find $CE : EF$.

Problem 11.7. Let $ABCD$ be a quadrilateral with an inscribed circle. Let M on AB, N on BC, P on CD, and Q on DA be the points of tangency of the quadrilateral with the circle. Suppose that $AM = a$, $BN = b$, $CP = c$, and $DQ = d$. Let Z be the intersection of MP and NQ. Find the ratio $MZ : ZP$.

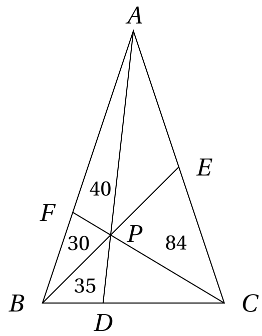


Problem 11.8. In $\triangle ABC$, the bisector of $\angle B$ intersects AC at D and intersects median AM at E. If $\sin \angle A = 4/5$ and $\sin \angle C = 24/25$, find $AE : EM$.

Problem 11.9. In $\triangle ABC$, angle bisectors AD and BE intersect at P. The sides of the triangle are $BC = 3$, $CA = 5$, $AB = 7$. If $BP = x$ and $PE = y$, compute the ratio $x : y$. (ARML)

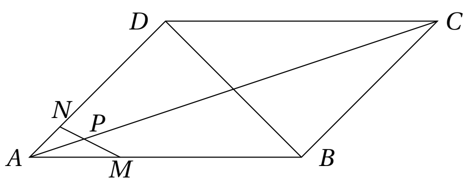


Problem 11.10. In $\triangle ABC$, cevians AD, BE, and CF intersect at point P. The areas of $\triangle PAF$, $\triangle PFB$, $\triangle PBD$, and $\triangle PCE$ are 40, 30, 35, and 84, respectively. Find the area of $\triangle ABC$. (AIME)

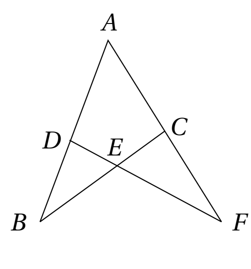


Problem 11.11. In $\triangle ABC$, $m\angle CBA = 72°$. E is the midpoint of side AC and D is a point on side BC such that $2BD = DC$; AD and BE intersect at F. Find the ratio of the area of $\triangle BDF$ to the area of quadrilateral $FDCE$. (AHSME)

Problem 11.12. In parallelogram $ABCD$, point M is on AB so that $\dfrac{AM}{AB} = \dfrac{17}{1000}$ and point N is on AD so that $\dfrac{AN}{AD} = \dfrac{17}{2009}$. Let P be the point of intersection of AC and MN. Find $\dfrac{AC}{AP}$. (AIME)



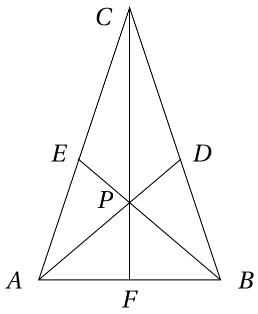
Problem 11.13. In $\triangle ABC$, D is on AB such that $AD : DB = 3:2$ and E is on BC such that $BE : EC = 3:2$. If ray DE and ray AC intersect at F, find $DE : EF$.



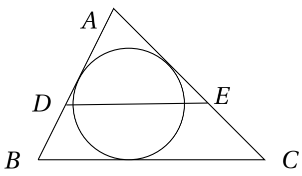
Problem 11.14. In $\triangle ABC$, A', B', and C' are on the sides BC, CA, and AB, respectively. Given that AA', BB', and CC' are concurrent at the point O, and that

$$\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92, \quad \text{find} \quad \frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}. \quad \text{(AIME)}$$

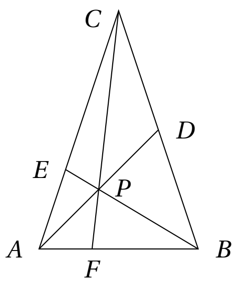
Problem 11.15. Let P be an interior point of $\triangle ABC$ and extend lines from the vertices through P to the opposite sides. Given that $AP = a$, $BP = b$, $CP = c$, $PF = PE = PD = d$, find the product abc if $a + b + c = 43$ and $d = 3$. (AIME)



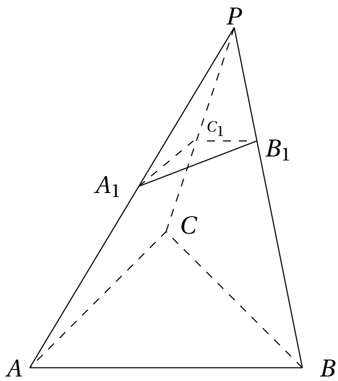
Problem 11.16. $\triangle ABC$ has $AB = 21$, $AC = 22$, and $BC = 20$. Points D and E are located on AB and AC, respectively, such that DE is parallel to BC and contains the center of the inscribed circle of $\triangle ABC$. DE can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. (AIME)



Problem 11.17. Point P is inside $\triangle ABC$. Line segments APD, BPE, and CPF are drawn with D on BC, E on AC, and F on AB. Given that $AP = 6$, $BP = 9$, $PD = 6$, $PE = 3$, and $CF = 20$, find the area of $\triangle ABC$. (AIME)



Problem 11.18. Let $PABC$ be a *right triangular pyramid*. That is, the pyramid's base $\triangle ABC$ is an equilateral triangle, and the altitude from P goes through the centroid M of $\triangle ABC$. A plane α intersects PA, PB, PC, and PM at points A_1, B_1, C_1, and M_1, respectively. If $PA_1 : A_1A = 2:3$, $PB_1 : B_1B = 3:2$, and $PC_1 : C_1C = 4:1$, find the ratio $PM_1 : M_1M$.



Problem 11.19. The base of a pyramid $SABCD$ is a parallelogram $ABCD$. A plane α intersects the sides SA, SB, SC, and SD at points A_1, B_1, C_1, and D_1, respectively. Given that

$$SA_1 = \frac{1}{3}SA,\ SB_1 = \frac{1}{5}SB, \text{ and } SC_1 = \frac{1}{4}SC,$$

find the ratio $\dfrac{SD_1}{SD}$.

Problem 11.20. Let A_1, B_1, C_1 be points on the sides of triangle ABC such that

$$\frac{BA_1}{BC} = \frac{CB_1}{CA} = \frac{AC_1}{AB} = \frac{1}{4}.$$

Let S be the area of triangle ABC. Find the area of the triangle PQR bounded by the lines AA_1, BB_1, and CC_1 in terms of S.