Generating Functions

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Abstract

Generating functions gives us a global perspective when we need to study a local property. We define generating functions and present its applications in algebra and combinatorics. We also introduce the **Snake Oil Method**, a powerful tool to evaluate combinatorial sums using generating functions.

Keywords: generating functions, generalized binomial theorem, combinatorics, snake oil method

1 Motivation

Why generating functions? Generating functions gives us a global perspective when we need to study a local property. It also gives us a simple way to encode and manipulate probabilities. It provides us a different angle and often helps us uncover the secrets that we were oblivious to previously. Quoting [3], "A generating function is a clothesline on which we hang up a sequence of numbers for display." Our first example illustrates the idea of using the coefficients of a polynomial to describe probabilities.

Example 1. *Compute the probability that you get exactly three 1's after rolling a standard 6-sided die five times. Then compute the probability that you get three 1's, one 2, and one 3 after rolling five times.*

Solution. For the first part, we see that rolling a 1 has a $\frac{1}{6}$ chance of occurring and any other number has a $\frac{5}{6}$ chance of coming up. There are $\binom{5}{3}$ ways to choose which three of the rolls in the sequence are 1's. So the total probability is $\binom{5}{3}(\frac{1}{6})^3(\frac{5}{6})^2$.

For the second part, there are $\binom{5}{3}$ ways to choose which three of the rolls are 1's, 2 ways to choose which of the remaining two rolls is a 2. The probability of any specific number coming up is again $\frac{1}{6}$. So the probability is $\binom{5}{3}\cdot 2\cdot(\frac{1}{6})^5$.

Now we illustrate how expanding a polynomial can help us solve this and similar problems. For the first part, we realize that there are two results we care about: rolling 1 or rolling anything else. To separate these probabilities in our polynomial, we make the coefficient of x be $\frac{1}{6}$ and the coefficient of y be $\frac{5}{6}$. Since the terms of the expansion should have total degree 5, we consider raising $(\frac{x}{6} + \frac{5y}{6})$ to the 5th power. Then the coefficient of x^3y^2 corresponds to the probability that rolling a die 5 times results in three 1's. Indeed, it is $\binom{5}{3}(\frac{1}{6})^3(\frac{5}{6})^2$, as computed earlier.

For the second part, there are three results we care about: rolling 1, rolling 2, or rolling 3. Then we consider the trinomial

$$(\frac{a}{6} + \frac{b}{6} + \frac{c}{6})^5.$$

The coefficient of a^3bc is $\binom{5}{3} \cdot 2 \cdot (\frac{1}{6})^5$, as predicted.

Similarly, as a polynomial with infinite degree, a generating function can encode information about a combinatorial sequence in its coefficients. Given a sequence $A = \{a_0, a_1, a_2, ...\}$, we define a **formal power series** $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Note that $G(x)$ is like a polynomial, except that it may have infinitely many terms. Note also that a_i, the ith member of A, is the coefficient of the term x^i of $G(x)$. We call $G(x)$ the **generating function** for the sequence $A = \{a_0, a_1, a_2, ...\}$.

Before we explore further, let's establish some useful identities.

2 Generalized Binomial Theorem

We all know the binomial theorem

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}x^{n-1} + x^n.$$

Its generalized form is quite useful when combined with generating functions.

Theorem 1 (Generalized Binomial Theorem)**.** *Let r be any real number. We have*

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n, \quad \textit{where} \quad \binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

When $r = -1$ in Theorem 1, we get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots .$$

We can also differentiate the geometric series repeatedly, and get

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \\
\frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} n x^{n-1}, \\
\frac{2}{(1-x)^3} &= \sum_{n=2}^{\infty} n(n-1) x^{n-2}, \\
&\cdots
\end{aligned}$$

Therefore,

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n.$$

We will use those identities in the next sections.

3 Counting

You might say that we do not have to use 5th degree polynomials to solve Example 1. That may be true, but generating functions can be used to solve complicated counting problems that are out of the reach of traditional methods. Let's look at an example.

Example 2. *In how many ways can we fill a bag with n fruits chosen from apples, bananas, oranges, and pears, subject to the following constraints?*

- *The number of apples must be even.*
- *There can be at most six bananas.*
- *The number of oranges is divisible by 7.*
- *There can be at most one pear.*

Solution. Let's use the method of divide and conquer and construct a generating function for selecting each type of fruit. First we select apples. We can select a set of 0 apples in 1 way, a set of 1 apple in 0 ways, a set of 2 apples in 1 way, and so on. Thus

$$A(x) = 1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}.$$

We can select a set of 0 bananas in one way, a set of 1 bananas in one way, ... Since there are at most six bananas, we have

$$B(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 = \frac{1-x^7}{1-x}.$$

The generating function for selecting oranges is similar to that for selecting apples, namely,

$$O(x) = 1 + x^7 + x^{14} + \cdots = \frac{1}{1-x^7}.$$

Finally, we can only select zero or one pear. So $P(x) = 1 + x$.
The generating function $G(x)$ for selecting from among all kinds of fruit is

$$G(x) = A(x)B(x)O(x)P(x) = \frac{1}{1-x^2}\frac{1-x^7}{1-x}\frac{1}{1-x^7}(1+x) = \frac{1}{(1-x)^2}.$$

Since

$$\begin{aligned} \frac{1}{(1-x)^2} &= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\ &= \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \cdots) \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots, \end{aligned}$$

the coefficient of x^n in $G(x)$ is $n+1$. Thus the number of ways to form a bag of n fruits is $\boxed{n+1}$.

Here is another example.

Example 3. *A set of playing cards has 32 cards in total. 30 of them are colored red, yellow, or blue, with 10 cards of each color, and given the numbers 1, 2, 3, ..., 10, respectively. The two remaining cards are both jokers and are designated with the number 0. A card with the number k counts for 2^k points. Let a good group be a group of cards where the sum of the points of the cards is 2017. Calculate the number of good groups.*

Solution. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$, where a_n is the number of groups of cards with sum of points equal to n. We see that

$$G(x) = (1+x)^2(1+x^2)^3(1+x^4)^3 \cdots (1+x^{1024})^3.$$

Multiplying both sides of the equation by $(1-x^2)^3$ and simplifying, we get

$$G(x) = \frac{(1+x)^2(1-x^{2048})^3}{(1-x^2)^3}.$$

From the last section, we know that

$$\frac{1}{(1-x^2)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^{2n}.$$

Since we are asked to find the coefficient of x^{2017}, we do not need to care about $(1-x^{2048})^3$. That is, we want the coefficient of x^{2017} in

$$\frac{(1+x)^2}{(1-x^2)^3} = (1+2x+x^2)\left(1 + \binom{3}{2}x^2 + \binom{4}{2}x^4 + \cdots\right).$$

The only way we can make a x^{2017} term is to pick the $2x$ term from $(1+x)^2$ and pick the $\binom{1010}{2}x^{2016}$ term from $(1-x^2)^{-3}$. Thus its coefficient is

$$2\binom{1010}{2} = \boxed{1019090}.$$

4 Recurrences

Generating functions are also very useful in solving recurrence relations, as we will see in the next example.

Example 4. *Given a sequence $A = \{a_0, a_1, a_2, ...\}$ such that*

$$a_0 = 0, \qquad a_1 = 1, \qquad \text{and} \qquad a_n = a_{n-1} + 2a_{n-2}$$

for all $n \geq 2$, find a closed form formula for a_n.

Solution. Computing the first few terms, we get $A = \{0, 1, 1, 3, 5, 11, ...\}$. Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence.

Now let's see where the given recurrence relation might lead us to. We rewrite $G(x)$ with $a_n = a_{n-1} + 2a_{n-2}$ in mind.

$$\begin{aligned} G(x) &= x + x^2 + 3x^3 + 5x^4 + 11x^5 + \cdots \\ xG(x) &= x^2 + x^3 + 3x^4 + 5x^5 + \cdots \\ 2x^2G(x) &= 2x^3 + 2x^4 + 6x^5 + \cdots \end{aligned}$$

Note that $xG(x) + 2x^2G(x) = x^2 + 3x^3 + 5x^4 + 11x^5 + \cdots$. Thus

$$G(x) - xG(x) - 2x^2G(x) = x \qquad \text{or} \qquad G(x) = \frac{x}{1 - x - 2x^2}.$$

Next we use **partial fractions** to make $G(x)$ more manageable. We get

$$G(x) = \frac{x}{1 - x - 2x^2} = \frac{1}{3(1-2x)} - \frac{1}{3(1+x)}.$$

Since

$$\begin{aligned} \frac{1}{1-2x} &= 1 + (2x) + (2x)^2 + (2x)^3 + \cdots, \\ \frac{1}{1+x} &= 1 + (-x) + (-x)^2 + (-x)^3 + \cdots, \end{aligned}$$

the nth term of our sequence A is

$$a_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n = \boxed{\frac{2^n - (-1)^n}{3}}.$$

Example 5. *Note that* $\frac{1}{89} = 0.011235955056....$ *Prove that*

$$\frac{1}{89} = \sum_{n=1}^{\infty} \frac{F_n}{10^{n+1}},$$

where F_n *is the* n*th term of the Fibonacci sequence defined by* $F_0 = 0$, $F_1 = 1$, *and* $F_{n+2} = F_n + F_{n+1}$ *for* $n \geq 2$.

Proof. Let $G(x) = \sum_{n=0}^{\infty} F_n x^n$ be the generating function for F_n. Similar to the solution to Example 4, we can get

$$G(x) = \frac{x}{1 - x - x^2} = F_0 + F_1x + F_2x^2 + F_3x^3 + \cdots. \qquad (*)$$

Plugging $x = \frac{1}{10}$ into $(*)$, we get

$$\frac{10}{89} = \frac{F_1}{10} + \frac{F_2}{100} + \frac{F_3}{1000} + \cdots,$$

and the result follows. □

5 Snake Oil Method

When each term a_n of a sequence $A = \{a_0, a_1, a_2, ...\}$ is itself a sum, generating function $G(x) = \sum_{n=0}^{\infty} a_n x^n$ is especially helpful. Then we have a double sum to work with and swapping the order of summation usually helps. Herbert Wilf [3] coined the name **Snake Oil Method**, a cure for everything. In his own words, "don't try to evaluate the sum that you're looking at. Instead, find the generating function for the whole parameterized family of them, then read off the coefficients." Let's look at an example.

Example 6. *For $n \geq 0$, compute*

$$\sum_{k\geq 0} \binom{k}{n-k}.$$

Solution. Here the free variable is n. So we define a generating function

$$G(x) = \sum_n \Big(\sum_{k\geq 0} \binom{k}{n-k} \Big) x^n.$$

Next we interchange the sums, and get

$$G(x) = \sum_{k\geq 0} \sum_n \binom{k}{n-k} x^n = \sum_{k\geq 0} x^k \sum_n \binom{k}{n-k} x^{n-k}.$$

Let $r = n - k$, we get

$$\begin{aligned} G(x) &= \sum_{k\geq 0} x^k \sum_r \binom{k}{r} x^r = \sum_{k\geq 0} x^k (1+x)^k \\ &= \sum_{k\geq 0} (x + x^2)^k = \frac{1}{1 - x - x^2}. \end{aligned}$$

From Example 5, we know that $\frac{x}{1-x-x^2}$ is the generating function for the Fibonacci sequence. Thus

$$\sum_{k\geq 0}\binom{k}{n-k} = F_{n+1} \qquad (n = 0, 1, 2, ...).$$

6 Exercises

1. Let $a_0 = 1$, $a_1 = 1$, and $a_n = 4a_{n-1} - 4a_n$ for $n \geq 2$. Use generating functions to find a formula for a_n in terms of n.

2. Find the number a_n of ways n dollars can be changed into 1-dollar or 2-dollar coins regardless of the order. For example, when $n = 3$, there are 2 ways, namely three 1-dollar coins or one 1-dollar coin and one 2-dollar coin.

3. Let n be a positive integer. Find the number a_n of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

4. Let a_0, a_1, a_2, ... be an increasing sequence of nonnegative integers such that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$, where i, j and k are not necessarily distinct. Determine a_{1998}.

5. Assume that for some positive integer n there are sequences of positive numbers a_1, a_2, ..., a_n and b_1, b_2, ..., b_n such that the sums

$$a_1 + a_2,\ a_1 + a_3,\ ...,\ a_{n-1} + a_n \qquad \text{and}$$
$$b_1 + b_2,\ b_1 + b_3,\ ...,\ b_{n-1} + b_n$$

   are same up to permutation. Prove that n is a power of two.

6. For $n \geq 0$, compute

$$\sum_{k\geq 0}\binom{n+k}{2k}2^{n-k}.$$

7. Prove that, for $m, n \geq 0$,

$$\sum_{k\geq 0}\binom{m}{k}\binom{n+k}{m} = \sum_{k\geq 0}\binom{m}{k}\binom{n}{k}2^k$$

   without evaluating either of the two sums.

References

[1] Evan Chen, Summation, `web.evanchen.cc/handouts/Summation/Summation.pdf`

[2] Albert Meyer and Ronitt Rubinfeld, Generating Functions, MIT 6.042J/18.062J

[3] Herbert Wilf, generatingfunctionology, `https://www.math.upenn.edu/~wilf/gfologyLinked2.pdf`